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# Thermally induced damped vibrations of an orthotropic rectangular plate of variable thickness

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**Abstract** The effect of damping with temperature variation on free axisymmetric vibrations of an orthotropic rectangular plate of linearly varying thickness has been analysed in present research work The governing differential equation of motion has been solved by Frobenius method The frequencies corresponding to the first two modes of vibrations have been obtained for an orthotropic rectangular plate with different combinations of boundary conditions for various values of damping constant and temperature gradient

**Keywords** : Orthotropic rectangular plate variable thickness damped vibration, temperature gradient

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## 1. Introduction

The present study is developed for the use of research workers in space technology, mechanical science and nuclear energy where certain components of the structures have to operate under elevated temperature and damping The engineering materials are of three types in terms of elastic symmetry *viz* isotropic, anisotropic and orthotropic type The isotropic material has an infinite number of symmetry every plane is plane of symmetry and it requires only two elastic constants for its characterization On the other hand, a material without any plane of symmetry is called fully anisotropic and it requires 21 independent elastic constants for its characterization Finally orthotropic materials are special case of anisotropic materials By definition, an orthotropic material has two orthogonal planes of symmetry where material properties are independent of direction within each plane, such materials required 9 independent elastic constants for their characterization. These studies has been intensively made by Timoshenko and Krieger [1] and Ghosh [2].

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Vibration analysis of orthotropic rectangular plate with variable thickness with different boundary conditions was studied by many authors [3–5]. Jorge [6] studied the vibrations of rectangular clamped plate using principle of virtual work. Kukla [7] has studied the frequency analysis of a rectangular plate with attached discrete systems Laura [10] gives the numerical analysis of flexural vibrations of rectangular plates having tapered thickness. Tomar and Gupta [11] have studied the thermal effect of frequencies of an orthotropic rectangular plate of linearly varying thickness. Biswas [12] has studied the thermaly induced vibrations of orthotropic rectangular plate resting on elastic foundation. Apply and Byers [13] have studied the fundamental frequency of simply supported rectangular plate with linearly varying thickness. Mc Nitt [14] introduced damping factor in free vibration of a damped elliptical plate.

The analysis presented here pertains to the effects of damping as well as temperature variation on frequencies of a rectangular orthotropic homogenous plate of linearly varying thickness with different boundary conditions. The first two modes of vibrations with clamped and simply supported edge conditions for various values of damping constant and the temperature gradient have been derived.

## 2. Theory and computation

The stress-strain relations in cartesian coordinate system for an orthotropic material are

$$\begin{aligned} e_x &= [\sigma_x - (\nu_{xy}\sigma_y + \nu_{xz}\sigma_z)]/E_x, \\ e_y &= [\sigma_y - (\nu_{yz}\sigma_z + \nu_{yx}\sigma_x)]/E_y, \\ e_z &= [\sigma_z - (\nu_{zx}\sigma_x + \nu_{zy}\sigma_y)]/E_z, \\ \gamma_{xy} &= \tau_{xy}/G_{xy}, \\ \gamma_{yz} &= \tau_{yz}/G_{yz}, \\ \gamma_{zx} &= \tau_{zx}/G_{zx}. \end{aligned} \tag{1 1}$$

In the case of plate (with $x$ and $y$ lying in the direction of orthotropy), the transverse stress $\sigma_z$, is assumed to be small relative to the inplane stresses, and so $\sigma_z = 0$

The transverse stress resultants $Q_x$ and $Q_y$ (per unit length) in terms of strain components are

$$Q_x = \int_{-h/2}^{+h/2} \tau_{xz}dz,$$

$$Q_y = \int_{-h/2}^{+h/2} \tau_{yz}dz, \tag{1.2}$$

where, $h$ is the thickness of the plate at any point. Similarly, relations between the moments [$M_x$, $M_y$, $M_{xy}$ (per unit length)] and the stress components are as follows

$$M_x = \int^{+h/2} \sigma_x z dz,$$

$$M_y = \int_{h/2}^{h/2} \sigma_y z dz,$$

$$M_{xy} = \int_{h/2}^{+h/2} \tau_{xy} z dz \tag{1 3}$$

Let $\bar{u}$, $\bar{v}$ and $\bar{w}$ be the displacements at a point $(x,y,z)$ in $x$, $y$ and $z$ directions respectively According to classical plate theory, the displacements ($u$, $v$ and $w$) of the plate at a point $(x,y,z)$ are approximated as

$$\bar{u} = u(x,y) - z\frac{\partial w}{\partial x},$$

$$\bar{v} = v(x,y) - z\frac{\partial w}{\partial y},$$

$$\bar{w} = w(x,y), \tag{1 4}$$

where $(u,v,w)$ are the displacements of the middle surface of the plate at the point $(x\,y,o)$ The equations (1 3) with the use of eq (1 4), take the form

$$Mx = -\left[D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right.$$

$$My = -\left[D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2}\right.$$

$$My = -2D_{xy}\frac{\partial^2 w}{\partial x \partial y},$$

$$Q_x = -\frac{\partial}{\partial x}\left[D_x \frac{\partial^2 w}{\partial x^2} + H\frac{\partial^2 w}{\partial y^2}\right],$$

$$Q_y = -\frac{\partial}{\partial x}\left[H\frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2}\right], \tag{1 5}$$

$D_x$ and $D_y$ are the flexural rigidity of the plate along $x$ and $y$-axis, respectively and $D_{xy}$ is called the torsional rigidity

Let us consider a plate element of length $dx$, breadth $dy$ and thickness $h$ and $(u,v,w)$ be the displacements of the middle plane at point $(x,y,o)$ at time $t$ The mass of the element is $\rho h\ dx\ dy$ where $\rho$ is the mass density per unit volume From Figures (1a) and (1b), it can easily be obtained that the equation of motion of the plate element in transverse direction is

$$\frac{\partial}{\partial x}Q_x dxdy + \frac{\partial}{\partial y}Q_y dxdy = \rho h dxdy\frac{\partial^2 w}{\partial t^2} \quad . \tag{1.6}$$

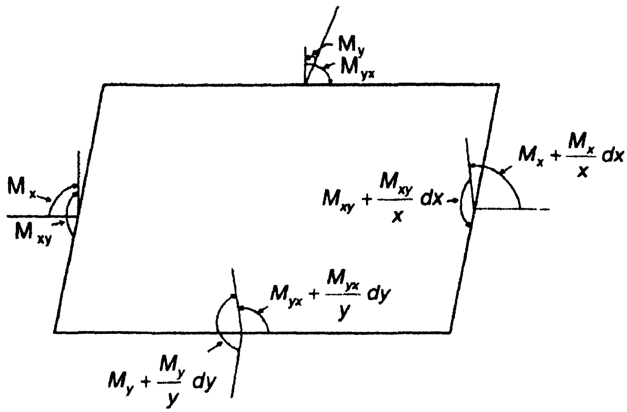


Figure 1(a). Moment resultants acting on the middle plane of a plate element

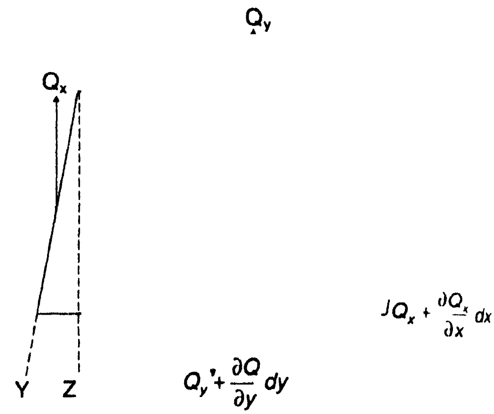


Figure 1(b). Shearing force resultants acting on the middle plane of a plate element

Taking moment of all the forces acting on the element with respect to the $x$-axis, one obtains

$$\frac{\partial}{\partial x} M_{xy} dxdy - \frac{\partial}{\partial y} M_y dxdy + Q_y dxdy = 0 \tag{1 7}$$

Similarly by taking moment with respect to the $y$-axis, one gets

$$\frac{\partial}{\partial y} M_{yx} + \frac{\partial}{\partial x} M_x - Q_x = 0 \tag{1 8}$$

Elimination of $Q_x$ and $Q_y$ from the eqs (1 6), (1 7) and (1 8) gives the differential equation of the transverse motion of the plate as

$$\frac{\partial^2}{\partial x^2} M_x + \frac{\partial^2}{\partial y^2} M_y + \frac{\partial^2}{\partial x \partial y} M_{yx} - \frac{\partial^2}{\partial x \partial y} M_{xy} = \rho h \frac{\partial^2 w}{\partial t^2} \tag{1 9}$$

It is worth noting that in case of orthotropic non-homogeneous plate of variable thickness with constant Poisson's ration, the Young's moduli, Shear modulus and thickness of plate in eqs (1.2) and (1.3) simply become variables and the relation (1 5) still apply Thus, substituting the values of $M_x$, $M_y$ and $M_{yx}$ from (1 5) in eqs (1 9) one get

$$D_x \frac{\partial^4 w}{\partial x^4} + D_y \frac{\partial^4 w}{\partial y^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial Y^2} + 2 \frac{\partial H}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2}$$

$$+ 2 \frac{\partial H}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y} + 2 \frac{\partial D_x}{\partial x} \frac{\partial^3 w}{\partial x^3} + 2 \frac{\partial D_y}{\partial y} \frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 D_x}{\partial x^2} + 4 \frac{\partial D_{xy}}{\partial x \partial y} \frac{\partial^3 D_{xy}}{\partial x \partial y}$$

$$+ \frac{\partial^2 D_y}{\partial y^2} \cdot \frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 D_1}{\partial x^2} \cdot \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 D_1}{\partial y^2} \cdot \frac{\partial^2 W}{\partial x^2} + 4 \frac{\partial^2 D_{xy}}{\partial x \partial y} \frac{\partial^2 W}{\partial x \partial y}$$

$$+ph\frac{\partial^2 W}{\partial t^2}+k\frac{\partial W}{\partial t}=0. \tag{1}$$

where $w$ is the transverse displacement.

It is assumed that the rectangular plate of orthotropic material is subjected to a steady one-dimensional temperature distribution along the length along $x$-axis.

$$T = T_0(1 - x) \tag{2}$$

where $T$ denotes the temperature excess above the reference temperature at any point at a distance $X = x/a$ and $T_0$ denotes the temperature excess above the reference temperature at the end $x = a$ or $x = 1$.

The temperature dependence of the modulus of elasticity for most of orthotropic materials can be expressed as

$$E_x(T) = E_1(1 - \gamma t)$$

and

$$E_y(T) = E_2(1 - \gamma t), \tag{3}$$

where $E_1$ and $E_2$ are the values of the modulus of elasticity respectively along $x$- and $y$-axis at the reference temperature *i.e.* at $T = 0$.

With the reference temperature taken as that at the end of the plate *i.e.* at $x = 1$, the modulus variations become

$$E_x(x) = E_1[1 - \alpha(1 - x)]$$

and

$$E_y(x) = E_2[1 - \alpha(1 - x)] \tag{4}$$

where $\alpha = \gamma T_0$ $(0 \leq \alpha < 1)$, a parameter known as temperature gradient.

It is assumed that the thickness varies in the $x$-axis only. Consequently, the thickness $h$, flexural rigidities $D_x$ and $D_y$ and torsional rigidity $D_{xy}$ of the plate become a function of $x$ only. Further, let the two opposite edges, $y = 0$ and $y = b$ of the plate be simply supported so that the plate undergoing free transverse vibrations with circular frequency $P$, may have solution of the following type

$$W(x,y,t) = \bar{w}(x)\sin(m\pi y/b)e^{\gamma t}\cos pt \tag{5}$$

where, $m$ is a positive integer.

Substitution of eq. (5) in (1) gives

$$\begin{aligned}&D_x\bar{w}_{,xxxx} + 2D_{x,x}\bar{w}_{,xxx} + [D_{x,xx} - 2Hr^2/a^2]\bar{w}_{,xx}\\ &- 2[v_yD_{x,x} + 2D_{xy,x}]r^2/a^2\,\bar{w}_{,x} + [D_yr^4/a^4 - r^2v_yD_{x,xx}]\bar{w}\\ &+ [-K^2/4\rho h - \rho hP^2]\bar{w} = 0\end{aligned} \tag{6}$$

where $r^2 = [m\pi a/b]^2$

Here, a comma followed by a suffix denotes partial differentiation with respect to that variable.

Thus, eq (6) reduces to a form independent of $y$ Now introducing the following non-dimensional variables,

$$\bar{H} = h/a,\ W = \bar{W}/a,\ X = x/a,\ D_x = D_x/a^3,\ D_y = D_y/a^3, \tag{7}$$

eq (6) becomes, in non-dimensional form

$$\begin{aligned}
&D_x W_{,xxxx} + 2D_{x,x} W_{,xxx} \\
&+[D_{x,}\mathbf{xx} - 2r^2(v_y D_x + D_{xy})]W_{,xx} \\
&= 2r^2[v_y D_{x,}\mathbf{x} + 2D_{xy,}\mathbf{x}]W_{,x} \\
&+ r^2[r^2 D_y - v_y D_{x/2,}\mathbf{xx}]W \\
&+ [-K^2/4\rho h - \rho a^2 \bar{H} P^2]W = 0
\end{aligned} \tag{8}$$

In view of the previous assumption, the present analysis is restricted to the modes having waves along $x$-axis only where the standing waves will be independent of the $y$-coordinate Further, the thickness varies linearly in $x$-axis only Therefore, let us assume

$$\bar{H}(x) = H_0(1 + \beta x), \tag{9}$$

where $\beta$ is the taper constant and $H_0 = \bar{H}/x = 0$ In view of eqs (3) and (9), the rigidity given by eq (7) becomes

$$D_x = D_0[1 - \alpha(1 - x)](1 + \beta x)^3$$

and

$$D_y = D_0[1 - \alpha(1 - x)](1 + \beta x)^3, \tag{10}$$

where,

$$D_0 = E_1 H_0^3/12(1 - v_x v_y)$$

and

$$\bar{D}_0 = E_2 H_0^3/12(1 - v_x v_y) \tag{11}$$

Upon substitution of eqs (9) and (10) into (8), the differential equation takes the following form

$$\begin{aligned}
&[1 - \alpha + \alpha x](1 + \beta x)^4 W_{,xxxx} \\
&+ 2[2\beta(1 + \beta x)^3(1 - \alpha + \alpha x) + \alpha(1 + \beta x)4]W_{,xxx} \\
&+ [3\alpha\beta(1 + \beta x)^3 + 3\beta\{\alpha(1 + \beta x)^3 + 2\beta(1 + \beta x)^2(1 - \alpha + \alpha x)\}]W_{,xx} \\
&+ 2r^2 v_y[(1 - \alpha + \alpha x)(1 + \beta x)^4]W_{,xx} \\
&+ 4r^2 v_y[\mathbf{G}^*(1 - v_x v_y))(1 + \beta x)^4]W_{,xx} \\
&- 2r^2 v_y[3\beta(1 + \beta x)^3(1 - \alpha + \alpha x) + \alpha(1 + \beta x)^4]W_{,x} \\
&- 4r^2[\mathbf{G}^*(1 - v_x v_y)^3\beta(1 + \beta x)^3]W_{,x} \\
&- r^4[\mathbf{E}^*(1 - \alpha + \alpha x)(1 + \beta x)^4]W \\
&- r^2 v_y[2\beta\{\alpha(1 + \beta x)^3 + 2\beta(1 + \beta x)^2(1 - \alpha + \alpha x)\} + 3\alpha\beta(1 + \beta x)^3]w \\
&- \mathbf{D}\mathbf{k}^2 r^{*2} W - \lambda^2(1 + \beta x)^2 w = 0
\end{aligned} \tag{12}$$

where

$$\lambda^2 = p^2 a^2 12(1 - v_x v_y) / E1^* H_0^2,$$
is a frequency parameter
$$D_k^2 = 3(1 - v_x v_y) K^2 / E_1 \rho, \quad I^* = 1 / H_0^2$$
$D_k$ is a damping factor
$$E^* = E_2 / E_1 \qquad G^* = G / E_1 \tag{13}$$
$$E1^* = E_1 / \rho$$

## 3. Solution

The differential eq (12) has been solved by the Frobenius method of Series Solution by assuming

$$W = \sum_{K=0}^{\infty} a_k x^{c+k}, \; a_0 \neq 0, \tag{14}$$

where $C$ is the exponent of singularity

The substitution in (12) gives

$$\begin{aligned}
&\sum_{k=0}^{\infty} a_k T_1^{(1)} b_k^{(3)} X^{c+k-4} \\
&+\sum_{k=0}^{\infty} a_k \left[ T_2^1 b_k^{(3)} + T_2^2 b_k^{(2)} \right] X^{c+k-3} \\
&+\sum_{k=0}^{\infty} a_k \left[ T_3^1 b_k^{(3)} + T_3^3 b_k^{(2)} + T_3^3 b_k^{(1)} \right] X^{c+k-2} \\
&+\sum_{k=0}^{\infty} a_k \left[ T_4^1 b_k^{(3)} + T_4^2 b_k^{(2)} + T_4^3 b_k^{(1)} \right] X^{c+k-1} \\
&+\sum_{k=0}^{\infty} a_k \left[ T_5^1 b_k^{(3)} + T_5^2 b_k^{(2)} + T_5^3 b_k^{(1)} + T_5^4 b_k + T_5^5 \right] X^{c+k} \\
&+\sum_{k=0}^{\infty} a_k \left[ T_6^1 b_k^{(3)} + T_6^2 b_k^{(1)} + T_6^3 b_k^{(1)} + T_6^4 b_k + T_6^5 \right] X^{c+k+1} \\
&+\sum_{k=0}^{\infty} a_k \left[ T_7^1 b_k^{(1)} + T_7^2 b_k + T_7^3 \right] X^{c+k+2} \\
&+\sum_{k=0}^{\infty} a_k \left[ T_8^1 b_k^{(1)} + T_8^2 b_k + T_8^3 \right] X^{c+k+3} \\
&+\sum_{k=0}^{\infty} a_k \left[ T_9^1 \right] X^{c+k+4} \\
&+\sum a_k \left[ T_{10}^1 \right] X^{c+k+5} = 0
\end{aligned} \tag{15}$$

The series expansion of eq (14) to be the solution, the coefficient of the powers of $X$ in eq (15) must be identically zero

Thus equating the coefficients of the lowest power of $X$ to zero, the following indicial roots are obtained

$$(1-\alpha)C(C-1)(C-2)(C-3)=0 \tag{16}$$

which gives $C = 0, 1, 2, 3$, since $\alpha < 1$

Now equating the coefficients of other powers of $X$, one finds that the constants $a_1$, $a_2$ and $a_3$ are indeterminate for $C = 0$ and hence, these can be taken as arbitrary constants along with $a_0$ The remaining constants $a_\lambda$ ($\lambda = 4, 5, 6$ ) are determined in terms of $a_0$, $a_1$, $a_2$ and $a_3$ from the recurrence relation

If the notations

$$a_K = a_0 f_K^{(0)} + a_1 f_K^{(1)} + a_2 f_K^{(2)} + a_3 f_K^{(3)}$$

$$K = 0, 1, 2, 3$$

are introduced in eq (14),

the solution for $W$, corresponding to $C = 0$, can be written as

$$W = a_0 F_0 + a_1 F_1 + a_2 F_2 + a_3 F_3 \tag{17}$$

where

$$\left.\begin{aligned} F_0 - F_0(x,\lambda^2) &= 1 + \sum_{k\ 4}^{\infty} f_k^{(0)} X^k, \\ F_1 = F_1(x,\lambda^2) &= X + \sum_{k\ 4}^{\infty} f_k^{(1)} X^k, \\ F_2 = F_2(x,\lambda^2) &= X^2 + \sum_{k\ 4}^{\infty} f_k^{(2)} X^k, \\ F_3 = F_3(x,\lambda^2) &= X^3 + \sum_{k\ 4}^{\infty} f_k^{(3)} X^k, \end{aligned}\right| \tag{18}$$

It is evident that no new solution will arise corresponding to other values of $C$ as it is already contained in the solution (17) with the arbitrary constants $a_0$, $a_1$, $a_2$ and $a_3$ Applying the test of Lamb [16] in the solution (17), it is convergent for $|\beta|<1$ and $|\alpha|<0\ 5$

## 4. Boundary conditions and frequency equations

The frequency equation for a rectangular plate with mixed clamped and simply supported boundary conditions can be obtained as follows

*[C-S-C-S-] plates* .

For a rectangular plate clamped at both the edges $x = 0$ and $x = 1$ (and simply supported at the remaining two edges), the deflection as well as the slope of the plate at $x = 0$ and $x = 1$ should be zero

$$W|_{x=0} = \partial W/\partial X|_{x=0} = 0, \quad W|_{x=1} = \partial W/\partial X|_{x=1} = 0 . \tag{19}$$

Applying the boundary conditions (19) to eq. (17), one gets the following equation after eliminating $a_0$, $a_1$, $a_2$ and $a_3$,

$$\begin{vmatrix} F_2(1,\lambda^2) & F_3(1,\lambda^2) \\ F_2'(1,\lambda^2) & F_3'(1,\lambda^2) \end{vmatrix} = 0. \tag{20}$$

*[C-S-S-S] plates* :

For rectangular plate clamped at $x = 0$ and simply supported at $x = 1$ (and simply supported at the remaining two edges),

$$W|_{x=0} = \partial W/\partial X|_{x=0} = 0, \quad W|_{x=1} = \partial^2 W/\partial X^2|_{x=1} = 0, \tag{21}$$

and hence the frequency equation for this plate, from eq. (17) comes out as

$$\begin{vmatrix} F_2(1,\lambda^2) & F_3(1,\lambda^2) \\ F_2''(1,\lambda^2) & F_3''(1,\lambda^2) \end{vmatrix} = 0. \tag{22}$$

*[S-S-C-S] plates* :

For a rectangular plate simply supported at $x = 0$ and clamped at $x = 1$ (and simply supported at the remaining two edges),

$$W|_{x=0} = \partial^2 W/\partial X^2|_{x=0} = 0, \quad W|_{x=1} = \partial W/\partial X|_{x=1} = 0, \tag{23}$$

and hence, the frequency equation for the S-S-C-S plate, from eq. (17) comes out as

$$\begin{vmatrix} F_1(1,\lambda^2) & F_3(1,\lambda^2) \\ F_1'(1,\lambda^2) & F_3'(1,\lambda^2) \end{vmatrix} = 0. \tag{24}$$

*[S-S-S-S] plates* :

For a rectangular plate simply supported at both the edges $x = 0$ and $x = 1$ (and also simply supported at the remaining two edges),

$$W|_{x=0} = \partial^2 W/\partial X^2|_{x=0} = 0, \quad W|_{x=1} = \partial^2 W/\partial X^2|_{x=1} = 0 , \tag{25}$$

and hence the frequency equation for this plate, from eq. (17) comes out as

$$\begin{vmatrix} F_1(1,\lambda^2) & F_3(1,\lambda^2) \\ F_1''(1,\lambda^2) & F_3''(1,\lambda^2) \end{vmatrix} = 0. \tag{26}$$

Here, dash denotes the differentiation with respect to $X$.

## 5. Results and discussion

Frequency eqs. (20), (22), (24) and (26) are transcendental equations in $\lambda$ from which infinite roots can be determined. The frequencies corresponding to the first two modes of vibrations of a C-S-C-S, C-S-S-S, S-S-C-S and S-S-S-S orthotropic rectangular plate have been computed for five values of damping constant $D_K$ and five values of temperature gradient ($\alpha$) and two values of taper constant ($\beta$ = 0.001, 0.003). The elastic constants for glass epoxy orthotropic materials have been used by Tomar and Gupta [11].

In computing the frequencies, terms whose absolute value is greater than $10^{-8}$ in the series for $W$ have been retained. From the graph, one concludes that the frequencies in first two modes of vibration decrease with increasing value of temperature gradient in all the four cases of different combinations of boundary conditions considered here. The frequencies corresponding to the first two modes of vibration increase with increasing value of damping constant ($D_K$). The frequency corresponding to the first two modes of vibration for various values of damping constant ($D_K$), temperature gradient ($\alpha$) for different values of taper constant and for various combinations of boundary conditions, are plotted in Figures 2, 3 and 4 For comparing the numerical values of

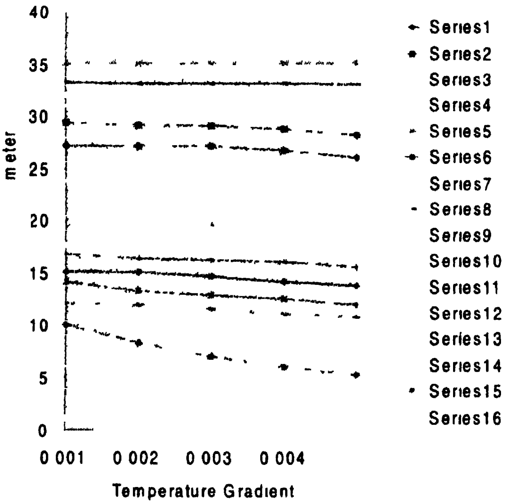


Figure 2. Variation of frequency parameter with temperature gradient for a rectangular plate of linearly varying thickness (colour on line).

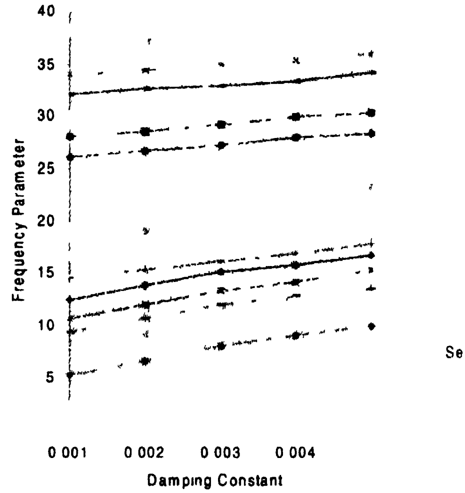


Figure 3. Variation of frequency parameter with damping constant for a rectangular plate of linearly varying thickness (colour on line)

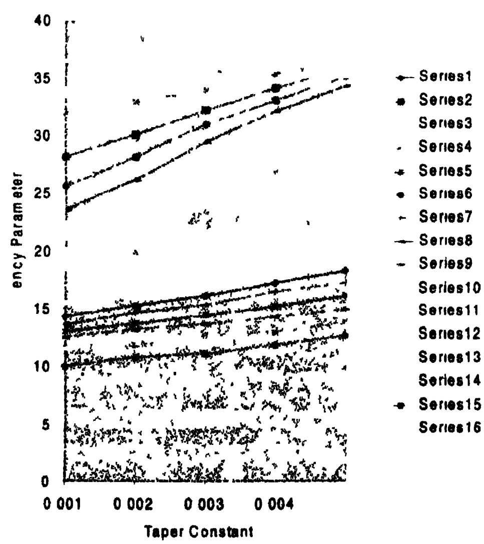


Figure 4. Variation of frequency parameter with taper constant for a rectangular plate of linearly varying thickness (colour on line).

the frequency with Tomar and Gupta [11], $\lambda$ has also been computed for the corresponding elastic constant for various boundary conditions of orthotropic rectangular plate and it is found that the results are in satisfactory agreement for the first two modes of vibration.

Normalised transverse deflection $W$ corresponding to first two modes of vibration for C-S-C-S and S-S-S-S plates at different points have been calculated for $a/b = 1.0$, $\alpha = 0.4$, $\beta = -0.3$ and $D_k = 0.001$. The results are plotted in Figure 5.

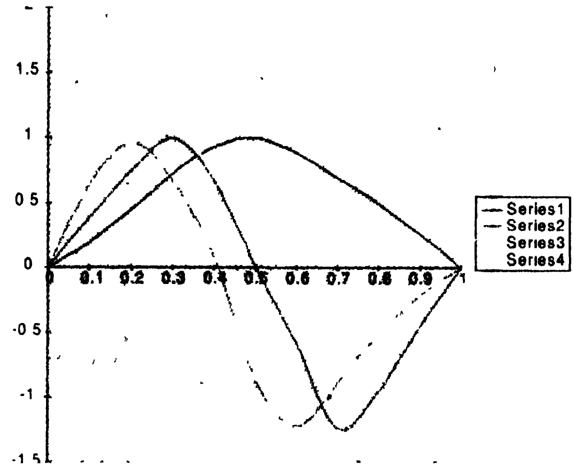


**Figure 5.** Transverse deflection of a rectangular plate of linearly varying thickness (colour on line).

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